MAXIMALLY FLAT ALLPASS FRACTIONAL HILBERT TRANSFORMERS

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ABSTRACT

Recently, a generalization of the Hilbert transformer, the fractional Hilbert transformer, was defined and developed. In this paper, we propose a design of the allpass filter to realize the fractional Hilbert transformer based on the maximally flat approximation to the desired phase response. The coefficients are solved analytically for the traditional Hilbert transformer which is a special case of the fractional Hilbert transformer. Based on the closed-form coefficients, we show that the maximally flat allpass Hilbert transformers are stable. Design examples indicate that the proposed filters exhibit good approximation to the desired frequency response.

1. INTRODUCTION

Hilbert transform (HT) is a basic and important tool in signal processing. In a communication system, it is used for single side-band modulation and then reduces the bandwidth needed for transmission. HT is also used for edge detection [1]. In [2], the HT is generalized into the fractional Hilbert transform (FHT) and two alternative definitions are given. One is a modification of the definition of HT and the frequency response is expressed by

$$H_{fht}(\omega) = \begin{cases} e^{-j\alpha\pi/2} & \text{for } \omega > 0 \\ e^{j\alpha\pi/2} & \text{for } \omega < 0 \end{cases} \tag{1}$$

Note that this FHT can be regarded as an allpass (AP) filter with a suitable specification of phase response. Another is implemented by using the above definition as well as the fractional Fourier transform to achieve a two-parameter FHT system. The discrete version of the FHT is proposed in [3].

Both FIR and IIR filters are investigated to realize the HT in extensive literature. In [4] and [5], the impulse responses of the FIR HTs are analytically solved in the maximally flat (MF) sense. A realization scheme based on decomposing the transfer function of HT into allpass subfilters is proposed in [6].

In this paper, the FHTs with MF phase response will be designed. The filter coefficients are obtained by solving linear equations. For the HT, a special case of the FHT, the coefficients are solved analytically. We show that the MF AP HTs are always stable by applying the Eneström-Kakeya theorem to the closed-form coefficients. The stability problem for the general FHTs is illustrated by numerically showing that the poles lie in the unit circle $|z| < 1$ for a certain α.

2. MAXIMALLY FLAT DESIGN OF AP FHT

The transfer function $H(z)$ of an Nth-order AP filter can be represented by

$$H(z) = z^{-N} \frac{\sum_{n=0}^{N} a_n z^n}{\sum_{n=0}^{N} a_n z^{-n}} = z^{-N} \frac{A(z^{-1})}{A(z)} \tag{2}$$

where the coefficients a_n's are real. Without loss of generality, we let $a_0 = 1$ to prevent from the null solution for a_n. The phase response $\theta(\omega)$ of $H(e^{j\omega})$ can be expressed by

$$\theta(\omega) = -N\omega + 2\tan^{-1} \frac{\sum_{n=0}^{N} a_n \sin n\omega}{\sum_{n=0}^{N} a_n \cos n\omega}. \tag{3}$$

Given the desired frequency response $\theta_d(\omega)$, we want to find a set of coefficients a_n's so that the phase error $\theta_e(\omega) = \theta_d(\omega) - \theta(\omega)$ is minimized. By expressing the desired phase response as $\theta_d(\omega) = -N\omega + \tilde{\theta}(\omega)$, the phase error $\theta_e(\omega)$ can be represented by

$$\theta_e(\omega) = \tilde{\theta}(\omega) - 2\tan^{-1} \frac{\sum_{n=0}^{N} a_n \sin n\omega}{\sum_{n=0}^{N} a_n \cos n\omega}. \tag{4}$$

However, it is difficult to minimize the above error function due to its nonlinearity. Therefore, we minimize an equivalent error function $e(\omega)$ which is defined by

$$e(\omega) = \tan \frac{1}{2}\tilde{\theta}(\omega) - \frac{\sum_{n=0}^{N} a_n \sin n\omega}{\sum_{n=0}^{N} a_n \cos n\omega}. \tag{5}$$

It is easy to show that $e(\omega) = n(\omega)/d(\omega)$ where

$$n(\omega) = \sum_{n=0}^{N} a_n \sin\left[\frac{1}{2}\tilde{\theta}(\omega) - n\omega\right] \tag{6}$$

$$d(\omega) = \cos \frac{1}{2}\tilde{\theta}(\omega) \sum_{n=0}^{N} a_n \cos n\omega. \tag{7}$$

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Finally, the numerator function $n(\omega)$ is the target error function to be minimized.

Let $\tilde{\theta}(\omega)$ in Eq.(6) be the ideal phase response expressed as

$$\tilde{\theta}(\omega) = \begin{cases} -\frac{1}{2}\alpha\pi, & \text{for } 0 < \omega < \pi; \\ \frac{1}{2}\alpha\pi, & \text{for } -\pi < \omega < 0 \end{cases} \quad (8)$$

where α is the desired fractional parameter. Putting the phase response $\tilde{\theta}(\omega)$ into Eq.(6), we can obtain the following equations

$$n(\omega) = -\sum_{n=0}^{N} a_n \sin\left(n\omega + \frac{1}{4}\alpha\pi\right), \text{ for } \omega > 0; \quad (9)$$

$$n(\omega) = \sum_{n=0}^{N} a_n \sin\left(-n\omega + \frac{1}{4}\alpha\pi\right), \text{ for } \omega < 0. \quad (10)$$

If $n(\omega)$ is minimized at $\omega = \omega_0$ in the MF sense, it has to satisfy the MF conditions of $d^k n(\omega)/d\omega^k = 0$ at $\omega = \pm\omega_0$. Because of the odd symmetry between Eq.(9) and (10), we can drop the case of $\omega_0 > 0$. Let $d^k n(\omega)/d\omega^k = 0$ at $\omega = \omega_0$ where $n(\omega)$ is expressed by Eq.(9). We have

$$\sum_{n=1}^{N} a_n n^k \sin\left(n\omega_0 + \frac{1}{2}k\pi + \frac{1}{4}\alpha\pi\right) = -\delta_k \sin\frac{1}{4}\alpha\pi \quad (11)$$

where δ_k is defined by

$$\delta_k = \begin{cases} 1, & \text{for } k = 0; \\ 0, & \text{otherwise.} \end{cases}$$

The coefficients a_n's can be solved by the N linear equations expressed in Eq.(11) for $k = 0, 1, \ldots, N-1$. In this paper, we choose the frequency $\omega_0 = \frac{1}{2}\pi$ to achieve the best approximation on the middle frequency range.

3. EXPLICIT SOLUTION FOR HILBERT TRANSFORMER

In the previous section, we show that the coefficients of the AP FHTs can be obtained by solving Eq.(11). The resulting FHTs exhibit the best approximation at $\omega = \frac{1}{2}\pi$ in the MF sense. However, these linear equations are ill-conditioned. For example, the reciprocal of the condition number is approximated to 1.281817×10^{-26} for the equations of Eq.(11) with $N = 20$ and $\alpha = \frac{1}{2}$ solved by MATLAB or similar software. That is, we can not solve Eq.(11) reliably for large N. The problem of numerical instability may be avoided by solving the equations analytically.

In this section, we will analytically solve the coefficients of MF AP FHTs for $\alpha = 1$ in Eq.(11). That is, we will find the closed-form solution of the AP MF HTs. Letting $\alpha = 1$ in Eq.(11), we can express the equations as

$$\sum_{n=1}^{N} a_n n^k \left(\cos\frac{1}{2}n\pi + \sin\frac{1}{2}n\pi\right) = -\delta_k, \text{ for even } k,$$
$$\sum_{n=1}^{N} a_n n^k \left(\cos\frac{1}{2}n\pi - \sin\frac{1}{2}n\pi\right) = 0, \text{ for odd } k. \quad (12)$$

The above equation can be solved by the ratio of two Vandermonde's determinants. If N is even, after some algebraic manipulations, we obtain the closed form of a_n's of

$$a_{2m} = \frac{\left(\frac{1}{2}\right)_m}{\left(M + \frac{1}{2}\right)_m}\binom{M}{m} \quad (13)$$

for $m = 0, 1, \ldots, M$, and

$$a_{2m+1} = -\frac{(M-m)\left(\frac{1}{2}\right)_m}{\left(M + m + \frac{1}{2}\right)\left(M + \frac{1}{2}\right)_m}\binom{M}{m} \quad (14)$$

for $m = 0, 1, \ldots, M-1$ where $M = N/2$, $\binom{M}{m}$ is the binomial coefficient, and $(x)_n$ is the Pochhammer's symbol defined by $(x)_0 = 1$ and $(x)_n = x \times (x+1) \times \cdots \times (x + n - 1)$. On the other hand, if $N = 2M + 1$, we have

$$a_{2m} = -a_{2m+1} = \frac{\left(\frac{1}{2}\right)_m}{\left(M + \frac{3}{2}\right)_m}\binom{M}{m} \quad (15)$$

for $m = 0, 1, \ldots, M$. Based on Eq.(15), it is obvious that $1 - z^{-1}$ is a factor of the denominators $A(z)$ of the odd-order MF AP HTs. Accordingly, we have the following property:

Property 1 *If N is an odd number and $M = (N-1)/2$, the denominator $A(z)$ of an Nth-order MF AP HT can be factored as $A(z) = (1 - z^{-1})\hat{A}(z^2)$ where*

$$\hat{A}(z) = \sum_{m=0}^{M} \hat{a}_m z^{-m}, \textit{and } \hat{a}_m = \frac{\left(\frac{1}{2}\right)_m}{\left(M + \frac{3}{2}\right)_m}\binom{M}{m}. \quad (16)$$

Remark. In[7] the authors analytically solved the coefficients of the MF AP orthonormal symmetric wavelet filters. These coefficients can be expressed by

$$b_m = (-1)^m \frac{(-M + K/4)_m}{(1 + K/4)_m}\binom{M}{m} \quad (17)$$

where K must be odd to satisfy the orthonormal condition. The odd-order MF AP HTs with coefficients expressed in Property 1 can be related with the MF AP orthonormal symmetric wavelet filters by $\hat{a}_m = (-1)^m b_m$ for $K = 4M + 2$.

Based on Eq.(1), one may synthesize the Nth-order FHTs by

$$H_\alpha(z) = \left(\cos\frac{1}{2}\alpha\pi\right) z^{-N} + \left(\sin\frac{1}{2}\alpha\pi\right) H_1(z) \quad (18)$$

where $H_1(z)$ is an Nth-order HT. That is, high order FHTs with flat phase response can be synthesized by the MF AP HTs without solving the ill-conditioned Eq.(11). We will prove that these FHTs are always stable by showing that the MF AP HTs are stable. However, the FHTs synthesized by AP HTs are not allpass filters. It is obvious that the FHT synthesized by Eq.(18) has the magnitude of $\cos \frac{1}{2}\alpha\pi + \sin \frac{1}{2}\alpha\pi$ at $\omega = 0$ if $H_1(z)$ is the MF AP HT. This DC value reaches its largest value of $\sqrt{2} > 1$ when $\alpha = 1/2$. Hence, a scaling factor is necessary to reduce this peak magnitude. A modified FHTs is proposed by $H'_\alpha(z) = kH_\alpha(z)$ where $k = (\cos \frac{1}{2}\alpha\pi + \sin \frac{1}{2}\alpha\pi)^{-1/2} = (1 + \sin \alpha\pi)^{-1/4}$.

4. STABILITY PROBLEM

To test the stability, one can apply the Schur-Cohn criterion or the more efficient Jury-Marden criterion [9] on the proposed AP filters. Thiran show that the MF AP fractional delay filters were stable by applying the Schur-Cohn criterion[8]. However, there exists other stability criteria suitable for the proposed filters without evaluating the Schur-Cohen determinants or establishing the Jury-Marden arrays. In this paper, we will apply the Eneström-Kakeya theorem [10] which is stated as

Theorem 1 *Let $p(x) = \sum_{n=0}^{N} a_n x^{N-n}$, $N \geq 1$, be a polynomial with $a_n > 0$ for $0 \leq n \leq N$. Let $r_n = a_{n+1}/a_n$ for $0 \leq n < N$. Then all the zeros of $p(x)$ are contained in the annulus*

$$\min_n r_n \leq |x| \leq \max_n r_n.$$

Based on the Eneström-Kakeya theorem, we have the following properties about the stability of the MF AP HTs.

Property 2 *All the poles of the MF AP HTs of even order are contained in the unit circle $|z| < 1$.*

Proof: Let $B(z) = A(-z)$ where $A(z)$ is the denominator of an even-order MF AP HT with coefficients expressed in Eqs.(13) and (14). Then all the coefficients of $B(z)$ satisfy

$$\begin{aligned} r_{2m} &= -\frac{a_{2m+1}}{a_{2m}} = \frac{2M - 2m}{2M + 2m + 1} < 1 \\ r_{2m-1} &= -\frac{a_{2m}}{a_{2m-1}} = \frac{2m - 1}{2m} < 1. \end{aligned}$$

According to the Eneström-Kakeya theorem, we conclude that all the zeros of $B(z)$ are contained in the unit circle $|z| < 1$ since the ratios of successive coefficients are less than unity. Then all the zeros of $A(z)$ are also in $|z| < 1$.

Property 3 *The MF AP HT of odd order has a pole at $z = 1$. All the other poles are contained in the unit circle $|z| < 1$.*

Proof: By applying the Eneström-Kakeya theorem to $\hat{A}(z)$ in Property 1, we have this property.

We can not conclude that the general MF AP FHTs are stable. However, by numerically computing the poles of the AP FHTs, the largest moduli of these poles are less than unity within a range of interest. Fig. 1 shows the plot of largest moduli for $2 \leq N \leq 16$ and $0 < \alpha < 1$.

5. DESIGN RESULTS

Fig. 2 and 3 show the design results of the MF AP FHTs . Fig. 2 is the plot of the phase responses of the 10th- and 11-order MF AP FHTs for $\alpha = 0.1, 0.3, 0.5, 0.7$ and 0.9. The phase responses are normalized by $2\{\arg[H(\omega)] + N\omega\}/\pi$. The purpose of the normalization is to find out the approximation to the desired α. There are bumps around $\omega = 0$ for odd-order filters. Figs. 3 and 4 show the magnitude and normalized phase responses for the FHTs synthesized by the 30th-order MF AP HT. We can not obtain these FHTs by solving Eq.(11) due to numerical instability. The phase responses shown in Fig. 3 exhibit good approximation to the desired phase responses within the middle frequency band. However, the magnitude responses can not remain unity over the whole band. Fig. 4 shows the magnitude responses which are scaled according the discussion in Section 3.

6. CONCLUSIONS

The MF AP FHTs are proposed in this paper. The coefficients of the FHTs are obtained by solving a set of linear equations. For the special cases of the HTs, the coefficients are solved analytically. Based on the closed-form expressions, we prove that the MF AP HTs are stable according to the Eneström-Kakeya theorem. The stability of the general MF AP FHTs is investigated by numerically calculating their poles for a certain range of α and N. The largest pole in modulus is less than unity for $N \leq 16$. However, since the general FHTs can be synthesized by the HTs, we show that IIR FHTs with flat phase response can be implemented by the MF AP HTs.

7. REFERENCES

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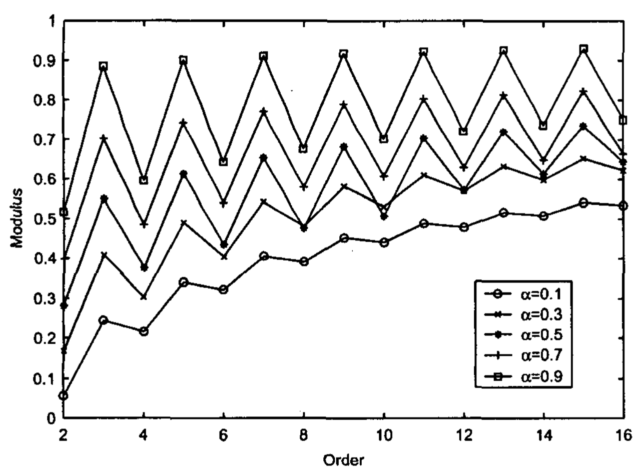

Fig. 1. The plot of the poles of largest modulus for $0 < \alpha < 1$ and $2 \leq N \leq 16$. These moduli are less than unity indicate the corresponding MF AP FHTs are stable.

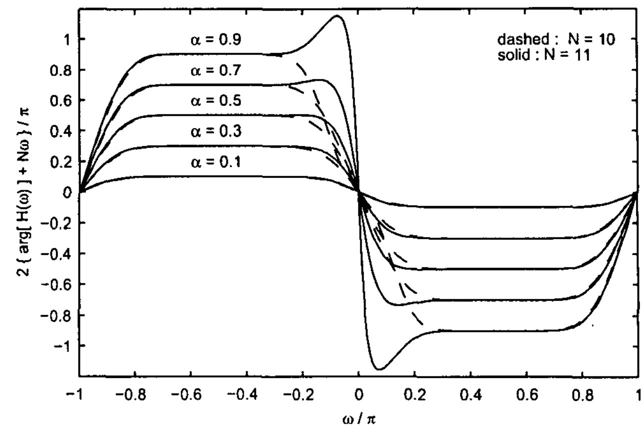

Fig. 2. The plot of normalized phase responses for $N = 10$ and 11.

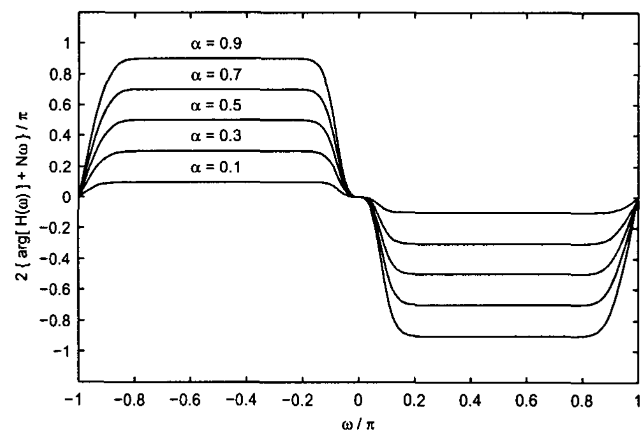

Fig. 3. The plot of normalized phase responses of 30th-order FHTs synthesized by the 30th-order MF AP HT.

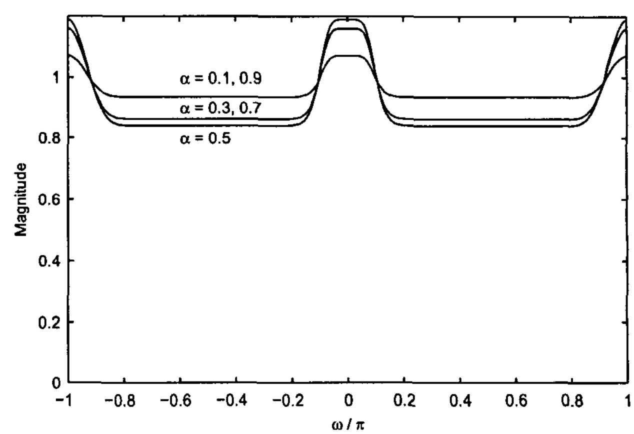

Fig. 4. The plot of magnitude responses of 30th-order FHTs synthesized by the 30th-order MF AP HT.